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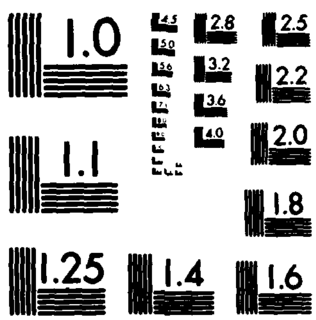

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TECHNICAL REPORT ARBRL-TR-02281

DIFFUSION EQUATION SOLUTION SEQUENCES

John F. Polk

January 1981

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (jfp)

A methodology is described for deriving expansions for the solutions of parabolic differential equations which are asymptotically valid for small times. Both the Cauchy problem and the first initial-boundary value problem are studied and particular attention is given to cases in which the data is discontinuous or incompatible with the equation. The basic procedure is called the DESS (Diffusion Equation Solution Sequence) method and all expansions are obtained in explicit form using certain special functions which are briefly discussed. Error estimates are rigorously stated but their proofs are not included in the present work.

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I. INTRODUCTION AND PRELIMINARIES

In this report we study the parabolic differential equation

$$D_t u = a(x)D_x^2 u + b(x)D_x u + c(x)u \qquad (1.1)$$

and describe a general method for constructing asymptotic expansions for its solutions, accurate for small times; both the Cauchy problem and the first initial-boundary value problem will be considered. The approximation method which we introduce will be referred to as the Diffusion Equation Solution Sequence or DESS method. It can be formulated whenever the coefficients a(x), b(x) and c(x) are sufficiently differentiable and the expansions can be explicitly determined when the initial and boundary data for the problem are expressed as the sum of smooth functions and jump functions (e.g., piecewise polynomials).

In his doctoral dissertation the author[1] conducted a rigorous mathematical analysis of the DESS method and established conditions for the validity of the expansions which it generates. The problem was formulated in this previous work in a somewhat different but equivalent fashion in which time was not assumed to be small but a small parameter, $\epsilon > o$, multiplied the right hand side of (1.1). The objective of the current report is to make the DESS method more accessible on the practical level by presenting its mechanics in a straightforward manner, stating the principal results concerning its validity without proof and avoiding the mathematical niceties as far as possible. Our approach will thus be a purely formal or "engineering" one in which (1) the expansions are sought in a certain form, (2) a necessary system of equations is derived for the individual terms of the expansions, and (3) a method of constructing explicit solutions for these equations is developed. It is the sequence of functions which results from this process that will be referred to as a DESS. In the present work we shall consider only the Cauchy problem and an initial-boundary value problem with Dirichelet (specified function values) boundary conditions. However, the method is more general and can also be used for Neumann's (flux specified) and Robin's (convective heat transfer) type of boundary data. The extension of steps (1) and (2) to such problems is very straightforward while step (3) would require more effort. In a forthcoming report we shall apply this technique to the problem of heat transfer in gun barrels.

[1] *J. F. Polk, "Asymptotic Expansions for the Solutions of Parabolic Differential Equations with a Small Parameter", Ph.D. Dissertation, Department of Mathematics, University of Delaware, Newark, DE, 1979.*

Let us now briefly outline the course which our investigation will follow. In Section II we attempt to construct formal asymptotic expansions for the solutions of (1.1) and obtain a system of equations for the individual terms of these expansions. Then in the next three sections we develop a number of special functions for the purpose of solving these equations. In Section III we recapitulate the functions H_γ and H^*_γ discussed in earlier work and introduce the new functions $H_{\gamma,n}$ and $H^*_{\gamma,n}$. Next in Section IV we define the functions $P_{\gamma,n}$ and $Q_{\gamma,n}$ which satisfy inhomogeneous forms of the diffusion equation. Then in Section V we formally define a DESS and construct as particular examples the sequences $\{E_{\gamma,n}\}$, $\{E^{\#}_{\gamma,n}\}$, $\{F_{\gamma,n}\}$ and $\{F^{\#}_{\gamma,n}\}$ whose terms are actually linear combinations of the functions $H_{\gamma,n}$. In Section VI we resume the discussion from Section II and write the desired asymptotic expansions in explicit form. Finally in Sections VII and VIII we put these expansions to use in constructing approximate solutions for the Cauchy and first initial-boundary value problem for equation (1.1).

This concludes our introductory remarks. Before proceeding to the main discussion we should first indicate some of the notational conventions which will be used in the following. From set theory we have

R = all real numbers

R^2 = two-dimensional space, coordinates x and t

(a,b) = $x \in R$ such that $a<x<b$

$[a,b]$ = $x \in R$ such that $a \le x \le b$

H = $R \times (o, \infty)$ = upper half plane, $t > o$

Q = first quadrant in $R^2 = (o, \infty) \times (o, \infty)$

Q^* = second quadrant in $R^2 = (-\infty, o) \times (o, \infty)$

The closure of a set in R or R^2 is indicated by an overbar, e.g.

$$\bar{H} = R \times [o, \infty)$$

$$\bar{Q} = [o, \infty) \times [o, \infty)$$

A compact set $S \subset R$ is a closed and bounded subset of R which, for practical purposes, may be considered to have the form $[a,b]$ with $a \le b$.

The derivatives of a function $f(x)$ will be denoted by $f^{(n)}(x)$ and the partial derivatives of a function $f(x,t)$ of two variables will be denoted

$$D_t f, \; D_x f, \; D_x^2 f$$

etc. A function $f(x)$ will be said to be of class $C^N(I)$ if it has N continuous derivatives in the open set I. The concept of Hölder continuity will be useful in stating some of the error bounds on our approximations but will not be essential for their derivation. A function $f(x)$ is said to be Hölder continuous (exponent α, $0 < \alpha < 1$) in an open set I if for each subset $S \subset I$ there exists a constant K such that

$$|f(x) - f(y)| \leq K|x-y|^{\alpha}$$

for all $x, y \in S$. This property is denoted by $f \in C^{\alpha}(I)$; for non-negative integers N we write $f \in C^{N+\alpha}(I)$ to indicate that $f^{(n)} \in C^{\alpha}(I)$ for each $n = 0,1,\ldots,N$. Readers not familiar with Hölder continuity need only consider the condition $f \in C^{N+\alpha}(I)$ as an intermediate property, stronger than $f \in C^N(I)$ but weaker than $f \in C^{N+1}(I)$.

In place of the usual Gamma function we have found it convenient to use the factorial notation

$$a! = \Gamma(a+1)$$

This is defined for $a > -1$ by

$$a! = \int_0^{\infty} t^a e^{-t}\, dt$$

and by the recursive formula

$$a! = \frac{(a+1)!}{(a+1)}$$

for $a < -1$, provided $a \neq -1, -2, -3, \ldots$ The factorial function becomes infinite for negative integers but its reciprocal is well defined (in fact, entire) for all choices of a, and in particular

$$\frac{1}{a!} = 0 \qquad a = -1, -2, \ldots \tag{1.2}$$

Finally, we wish to note two algebraic identities which will be used in Section V:

$$\sum_{k=0}^{n} \sum_{j=0}^{k} a_{k,j} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_{i+j,j} \tag{1.3}$$

and

$$\sum_{j=1}^{n} \sum_{i=0}^{2(n-j)} a_{ij} = \sum_{k=1}^{n} \sum_{j=1}^{k} a_{k-j,j} + \sum_{k=n+1}^{2n-1} \sum_{j=1}^{2n-k} a_{k-j,j} \tag{1.4}$$

A proof of these identities is given in the introduction to reference 1 and will not be included here. If interested, the reader can also verify these by writing a few terms of the summands in a geometric array and observing how the different sides of the identities are obtained by summing the array elements in different orders.

II. FORMAL EXPANSION PROCEDURES

The parabolic differential equation

$$D_t u = Lu \equiv a(x)D_x^2 u + b(x)D_x u + c(x)u \tag{2.1}$$

is of fundamental importance to transport processes. The unknown function $u(x,t)$ can be thought of as any transport quantity, such as temperature, species concentration, vorticity, etc. and the terms on the right hand side of (2.1) correspond to contributions to its rate of change due to diffusion, convection (advection) and species production (or consumption) respectively. In the present form the coefficients depend only on the spatial coordinate x and thus equation (2.1) is linear although it is frequently non-linear in actuality. Nevertheless this equation is pertinent to non-linear problems for two reasons: (1) a proper understanding of non-linear processes can only be achieved after a thorough rendering of the linear case, and (2) a non-linear equation can be solved numerically by quasi-linearization in which non-linear coefficients are expressed as functions of the spatial variable only during a single time step. This procedure is valid for sufficiently short time intervals and can be performed repeatedly to advance the computation over longer time spans. In computational fluid dynamics even simpler versions of (2.1) are sometimes studied[2] to gain insight into the nature of transport processes. In fact, it seems that the majority of the working tools used in practice (stability criteria, order of magnitude error estimates, etc.) have been rigorously justified only in the context of linear initial value problems.

In order to motivate more complicated expansion procedures for obtaining approximate solutions of (2.1) let us first introduce the simplest, most direct method. We suppose that initial data has been specified in the form

$$u(x,0) = f(x) \tag{2.2}$$

valid for x in a given interval $I \subset R$. The N-term Taylor expansion for u in the time direction from the initial line is then

[2] *P. J. Roache, "Computational Fluid Dynamics", 2nd Edition, Hermosa Publishers, Albuquerque, NM, 1976.*

$$\sum_{k=0}^{N} D_t^{(k)} u(x,0)\ t^k/k!$$

But, by repeated formal substitution of (2.1) this can be written as

$$\sum_{k=0}^{N} L^{(k)} u(x,0)\ t^k/k!$$

or

$$\sum_{k=0}^{N} L^{(k)} f(x)\ t^k/k!$$

where $L^{(k)}$ denotes the operator L applied k times. Because of its special significance we shall refer to the last expression as the N-term regular expansion for u and denote it by

$$U^N(x,t) \equiv \sum_{k=0}^{N} L^{(k)} f(x)\ t^k/k! \qquad (2.3)$$

This expansion is well defined if all of the terms $L^{(k)} f(x)$ are meaningful; this holds when a, b, and c have $2N-2$ derivatives and f has 2N derivatives at x. Using slightly stronger assumptions than these, the regular expansion has been shown to accurately approximate u and we shall state these results formally in Sections VII and VIII.

Assuming that a, b and c are suitably behaved, the regular expansion breaks down in two not infrequent situations:

(1) The initial value function f(x) is not sufficiently smooth and thus U is not even defined. Since f(x) is usually determined by physical measurement or inference, there is no guarantee that it will have any particular "analytic" form. For example, bringing both hot and cold objects suddenly together leads to discontinuous initial values. On the other hand it can be supposed that almost any physically reasonable f(x) is at least piecewise smooth with jumps of various orders occuring at discrete points. For this reason a particularly relevant choice of initial values is the jump function

$$f(x) = h_\gamma(x-x_o) \qquad (2.4)$$

where

$$h_\gamma(z) \equiv \begin{cases} z^\gamma/\gamma! & z > 0 \\ 0 & z \leq 0 \end{cases} \qquad (2.5)$$

with $\gamma \geq 0$.

(2) a boundary condition can be imposed along a boundary such as $x=x_o$ which is incompatible with the regular expansion even when well-defined. In fact, the most that can be expected in general is that the boundary condition is continuous with the initial data at the corner point (x_o,o). This would ensure that only the lowest order, but no higher term of the regular expansion agrees with the prescribed boundary data. Such incompatibilities are evidenced by "thermal boundary layers" in short term heat conduction problems. The most fundamental boundary condition is that in which the function values are specified (Dirichelet condition) along a fixed boundary and, among these, the most important are those which are homogeneous or can be expressed as a power of t.

From these considerations we are led to formulate, as "canonical" problems, equation (2.1) together with the following choices of initial/boundary data

$(IVP)_o$	$u(x,o) = h_\gamma\ (x-x_o)$	$x \in R$
$(BVP)_o$	$u(x,o) = 0$	$x \leqq x_o$
	$u(x_o,t) = h_{\gamma/2}\ (t)$	$o<t\leq \Delta t$

Where $\gamma\geq 0$ and $\Delta t>0$. (The use of $\gamma/2$ rather than γ in $(BVP)_o$ leads to more conveniently stated results.) The upper bound on time is indicated by Δt to emphasize our interest in short duration approximations. Let us also formulate the two converse problems

$(IVP)^{\#}_o$	$u(x,o) = h^*_\gamma\ (x-x_o)$	$x \in R$
where	$h^*_\gamma(z) = h_\gamma\ (-z)$, and	
$(BVP)^{\#}_o$	$u(x,o) = 0$	$x\geq x_o$
	$u(x_o,t) = h_{\gamma/2}\ (t)$	$o<t\leq \Delta t$

The expansions for these latter two problems can be obtained from the **preceding** ones by a straightforward transformation. We shall state the parallel results for convenience without additional comment.

From the theory of parabolic differential equations[3,4,5] it is

[3] *A. Friedman, "Partial Differential Equations of Parabolic Type", Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.*
[4] *O. A. Ladyzenskaja, V. A. Salonnikov, N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type", American Mathematical Society Translations of Mathematical Monographs, Volume 23, 1968.*
[5] *P. C. Rosenbloom, "Linear Partial Differential Equations", Surveys in Applied Mathematics V, John Wiley and Sons, Inc., New York, 1958.*

well known that a solution $u(x,t)$ exists for each of the problems just formulated; furthermore it will be unique if restricted to the class of functions such that

$$|u(x,t)| \leq K \exp(k x^2) \tag{2.6}$$

for some constants K, $k \geq 0$, uniformly for x,t in the domain of interest.

Let us now proceed with the formal development of asymptotic expansions for $u(x,t)$ basing our derivation on two considerations: first our interest is primarily in the local region near $x=x_o$ (since the regular expansion suffices elsewhere) and second we are concerned only with the short time behavior. We shall suppose that equation (2.1) has already been written in a form such that u, x and t are non-dimensional with t restricted to the interval $o \leq t \leq \Delta t$ where $\Delta t << 1$.

To emphasize the local short term behavior of u let us introduce the following "stretched" variables

$$\begin{aligned} \sigma &= (x-x_o)/\varepsilon \\ \tau &= t/\varepsilon^2 \end{aligned} \tag{2.7}$$

where, for convenience, we henceforth let $\varepsilon = \sqrt{\Delta t}$.
In terms of these variables we may re-express the solution u as the transformed function $\tilde{u}$ defined by

$$\tilde{u}(\sigma,\tau) = u(x_o + \varepsilon\sigma, \varepsilon^2\tau) = u(x,t)$$

Similarly the coefficients a, b, and c may be expressed as

$$\begin{aligned} \tilde{a}(\sigma) &= a(x_o + \varepsilon\sigma) \\ \tilde{b}(\sigma) &= b(x_o + \varepsilon\sigma) \\ \tilde{c}(\sigma) &= c(x_o + \varepsilon\sigma) \end{aligned}$$

Substitution of these into (2.1) leads to the following equation for $\tilde{u}$

$$D_\tau \tilde{u} = \tilde{a}(\sigma) D_\sigma^2 \tilde{u} + \varepsilon\, \tilde{b}(\sigma) D_\sigma \tilde{u} + \varepsilon^2\, \tilde{c}(\sigma) \tilde{u} \tag{2.8}$$

In this form standard perturbation methods can be employed to develop expansions for $\tilde{u}$ in terms of the small parameter $\sqrt{\Delta t}$. Let us suppose that u can be expanded in the form

$$\tilde{u} \sim \varepsilon^p [\tilde{u}_o + \varepsilon\, \tilde{u}_1 + \varepsilon^2 \tilde{u}_2 + \ldots \tag{2.9}$$

where p is not yet specified but depends on the particular choice of boundary/initial data. If the coefficients a, b and c are

sufficiently differentiable at $x=x_o$ then we can form their n-term Taylor expansions. In terms of the stretched variable σ these are

$$\tilde{a}(\sigma) = \sum_{k=0}^{n} a_k(\varepsilon\sigma)^k/k! + A_n$$

$$\tilde{b}(\sigma) = \sum_{k=0}^{n} b_k(\varepsilon\sigma)^k/k! + B_n$$

$$\tilde{c}(\sigma) = \sum_{k=0}^{n} c_k(\varepsilon\sigma)^k/k! + C_n$$

where a_k, b_k and c_k denote the Taylor coefficients

$$\begin{aligned} a_k &= a^{(k)}(x_o) \\ b_k &= b^{(k)}(x_o) \\ c_k &= c^{(k)}(x_o) \end{aligned} \tag{2.10}$$

and A_n, B_n and C_n are remainder terms. If (2.9) - (2.10) are substituted into (2.8) and terms with similar powers of ε collected together then the following system of equations results

$$[D_\tau - a_o D_\sigma^2]\,\tilde{u}_o = 0$$

$$[D_\tau - a_o D_\sigma^2]\,\tilde{u}_1 = \tilde{L}_1\tilde{u}_0$$

$$[D_\tau - a_o D_\sigma^2]\,\tilde{u}_2 = \tilde{L}_1\tilde{u}_1 + L_2\tilde{u}_0$$

and in general

$$[D_\tau - a_o D_\sigma^2]\,\tilde{u}_n = \sum_{k=1}^{n} \tilde{L}_k\,\tilde{u}_{n-k} \tag{2.11}$$

for n=1, 2, ... where

$$\tilde{L}_k\tilde{u} = a_k\frac{\sigma^k}{k!}D_\sigma^2\tilde{u} + b_{k-1}\frac{\sigma^{k-1}}{(k-1)!}D_\sigma\tilde{u} + c_{k-2}\frac{\sigma^{k-2}}{(k-2)!}\tilde{u}.$$

(In this notation we are making use of (1.2) in the case k=1.). The operator $\tilde{L}_k$ is clearly well defined whenever a, b, and c have k, k-1 and k-2 derivatives at x_o, respectively.

Next, by rewriting the boundary and initial data for $(IVP)_o$ and $(BVP)_o$ in terms of the stretched variables we obtain

$$\tilde{u}(\sigma,o) = \varepsilon^\gamma h_\gamma(\sigma) \qquad \sigma\in R$$

for problem $(IVP)_o$, and

$$\tilde{u}(\sigma,o) = 0 \qquad \sigma\leq o$$

$$\tilde{u}(o,\tau) = h_{\gamma/2}(\tau) \qquad o \leq \tau \leq 1$$

for problem $(BVP)_o$. From this it follows that

$$p = \begin{cases} \gamma \text{ in problem } (IVP)_o \\ 0 \text{ in problem } (BVP)_o \end{cases} \tag{2.12}$$

and that the individual terms of the expansion for $\tilde{u}$ must satisfy the supplementary conditions

$$\tilde{u}_k(\sigma,o) = \begin{cases} h_\gamma(\sigma) & \text{for } k=0 \\ 0 & \text{for } k=1,\ 2,\ \ldots \end{cases} \tag{2.13}$$

for problem $(IVP)_o$ and

$$\tilde{u}_k(\sigma,0) = 0 \qquad \text{for } k=0,1,2\ldots \tag{2.14}$$

$$\tilde{u}_k(o,\tau) = \begin{cases} h_{\gamma/2}(\tau) & \text{for } k=0 \\ 0 & \text{for } k=1,2,\ldots \end{cases} \tag{2.15}$$

with $\sigma,\ \tau \geq 0$ for problem $(BVP)_o$.

The forgoing systems of equations and initial/boundary conditions for the terms $\hat{u}_k$ have a unique solution among the class of functions satisfying (2.6) which can be uniquely determined. To accomplish this we suspend our present discussion for the moment and in the following three sections investigate some relevant special functions. Readers not interested in the details can continue in Section VI.

III. THE FUNCTIONS H_γ AND $H_{\gamma,n}$

The functions H_γ have been discussed in previous work[1,6]. For convenience we briefly recall their definitions and basic properties. When $\gamma > -1$ we define

$$H_\gamma(x,t) \equiv (4\pi t)^{-1/2} \int_0^\infty (s^\gamma/\gamma!)\ \exp\ (-(x-s)^2/4t)ds \tag{3.1}$$

for $t>0$; for $\gamma \leq -1$ H_γ is defined recursively by

$$H_\gamma(x,t) \equiv D_x\ H_{\gamma+1}(x,t) \tag{3.2}$$

for $t>0$. (This also holds as an identity for $\gamma > -1$.) Along $x=0$ these functions are assigned the values

$$H_\gamma(x,o) = h_\gamma(x) \qquad x \in R \tag{3.3}$$

where $h_\gamma(x)$ is defined by (2.5); except at the origin $x=t=0$ with $\gamma \leq 0$ this can be seen[1] to form a continuous extension of (3.1) and (3.2). These functions satisfy the relationships

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[6] *J. F. Polk, "Special Function Solutions of the Diffusion Equation", ARBRL-TR-02182, US Army Ballistic Research Laboratory, 1979.* (AD #A075324)

$$\gamma H_\gamma(x,t) = x\, H_{\gamma-1}\,(x,t) + 2t\, H_{\gamma-2}\,(x,t) \tag{3.4}$$

and

$$H_\gamma(x,t) = \sqrt{a}^{\,-\gamma}\, H_\gamma\,(x\sqrt{a},\ a\,t) \tag{3.5}$$

for any $\gamma \in R$ and $a > 0$. Particular examples of H_γ functions are

$$H_o(x,t) = (1/2)\ \text{erfc}\ (-x/\sqrt{4t})$$

$$H_{-1}(x,t) = (4\pi t)^{-1/2} \exp\ (-x^2/4t) \tag{3.6}$$

$$H_{-2}(x,t) = -(x/2t)\ H_{-1}\ (x,t)$$

where

$$\text{erfc}(z) = (4/\pi)^{1/2} \int_z^\infty \exp\ (-s^2)ds$$

The reflected functions h^*_γ and H^*_γ are defined by

$$h^*_\gamma\ (z) = h_\gamma(-z) \tag{3.7}$$

$$H^*_\gamma\ (x,t) = H_\gamma\ (-x,t) \tag{3.8}$$

All of the functions H_γ and H^*_γ were seen to be infinitely differentiable with respect to x and t for $t > 0$ and to satisfy the formulas

$$D_x\ H_\gamma = H_{\gamma-1} \qquad D_t\ H_\gamma = H_{\gamma-2} \tag{3.9}$$

$$D_x\ H^*_\gamma = -H^*_{\gamma-1} \qquad D_t\ H^*_\gamma = H^*_{\gamma-2} \tag{3.9*}$$

Thus they are solutions of the heat or diffusion equation

$$[D_t - D_x^2]\ u = 0 \tag{3.10}$$

for $t>0$. Along $x=0$ they assume the values

$$H_\gamma(o,t) = H^*_\gamma\ (o,t) = \sqrt{t}^{\,\gamma}/2(\gamma/2)! \tag{3.11}$$

Let us now define

$$H_{\gamma,n}\ (x,t) = \frac{x^n}{n!}\ H_\gamma\ (x,t) \tag{3.12}$$

and

$$H^*_{\gamma,n}\ (x,t) = H_{\gamma,n}\ (-x,t) = \frac{(-x)^n}{n!}\ H_\gamma\ (-x,t) \tag{3.12*}$$

where $\gamma \in R$ and n is an integer. These can be regarded as generalizations of the functions H_γ since setting $n=0$ yields

$$H_{\gamma,o}(x,t) = H_\gamma(x,t) \tag{3.13}$$

$$H^*_{\gamma,o}(x,t) = H^*_\gamma(x,t) \tag{3.13}*$$

Moreover, for negative integers n, these functions are seen to vanish identically

$$H_{\gamma,n} = H^*_{\gamma,n} = 0 \tag{3.14}$$

n=-1,-2,-3, ... , in view of (1.2). Using (3.3) and (3.11) we see that along t=0 and x=0 these functions take on the values

$$H_{\gamma,n}(x,o) = h_\gamma(x)\, h_n(x) = \begin{cases} \dfrac{x^{\gamma+n}}{\gamma!\; n!} & x>0 \\ 0 & x\leq 0 \end{cases} \tag{3.15}$$

$$H_{\gamma,n}(o,t) \begin{cases} \tfrac{1}{2}h_{\frac{1}{2}\gamma}(t) & \text{if } n=0 \\ 0 & \text{if otherwise} \end{cases} \tag{3.16}$$

From differentiation formulae (3.9) and Leibnitz's rule for repeated differentiation of a product we obtain

$$D_x H_{\gamma,n} = H_{\gamma,n-1} + H_{\gamma-1,n} \tag{3.17}$$

$$D_x^k H_{\gamma,n} = \sum_{j=0}^{k} \binom{k}{j} H_{\gamma-j,n-k+j} \tag{3.18}$$

$$D_t H_{\gamma,n} = H_{\gamma-2,n} \tag{3.19}$$

and

$$[D_x^2 - D_t]\, H_{\gamma,n} = 2\, H_{\gamma-1,n-1} + H_{\gamma,n-2} \tag{3.20}$$

For any constant $\alpha>0$ it then follows that $H_{\gamma,n}(x,at)$ satisfies the following inhomogeneous form of the diffusion equation

$$[aD_x^2-D_t]\, H_{\gamma,n}(x,at) = 2aH_{\gamma-1,n-1}(x,at) + aH_{\gamma,n-2}(x,at) \tag{3.21}$$

for t>0. Also, for any a>0, we can obtain the following identity from (3.5)

$$H_{\gamma,n}(x,t) = a^{-(\gamma+n)/2}\, H_\gamma(x\sqrt{a},at) \tag{3.22}$$

In Reference 1 these functions were studied in greater detail, in particular, their asymptotic behavior for large $|x|$ was characterized. In the present work we do not require this analysis except to recall that

$$|H_{\gamma,n}(x,t)| \leq \text{const.}\ [h_{\gamma+n}(x) + \sqrt{t}^{\,\gamma+n}] \tag{3.23}$$

and thus $H_{\gamma,n}$ clearly belongs to the class of functions satisfying (2.6).

IV. THE FUNCTIONS $P_{\gamma,n}$ AND $Q_{\gamma,n}$

As the next step toward solving the equations derived in Section II we consider the functions $P_{\gamma,n}$ and $Q_{\gamma,n}$. These are defined for any $\gamma \epsilon R$ and any integer n by

$$Q_{\gamma,n}(x,t) = \sum_{k=1}^{n+1} (-1/2)^{n+2-k} H_{\gamma+n+2-k,k}(x,t) \tag{4.1}$$

and

$$P_{\gamma,n}(x,t) = Q_{\gamma,n}(x,t) - c_{\gamma,n} H_{\gamma+n+2}(x,t) \tag{4.2}$$

where

$$c_{\gamma,n} = \sum_{k=1}^{n+1} (-1/2)^{n+2-k} \binom{\gamma+n+2}{k}. \tag{4.3}$$

In the last expression we have used the binomial coefficient notation

$$\binom{a}{k} \equiv \frac{a\ (a-1)\ (a-2)\ .\ .\ .\ (a-k+1)}{k!}$$

These functions are thus linear combinations of the $H_{\gamma,n}$ functions. Their significance is indicated in the following two propositions. Recall our notation Q and Q^* for the first and second quadrants in R^2.

[redacted] If $\gamma+n>-2$ then (4.1) defines the unique function which is continuous in $\bar{Q}^*$, uniformly bounded whenever t is bounded, which satisfies the equation

$$[D_t - D_x^2]\ Q_{\gamma,n}(x,t) = H_{\gamma,n}(x,t) \tag{4.4}$$

in $\bar{Q}^*$ and takes on the homogeneous boundary and initial values

$$Q_{\gamma,n}(x,o) = 0 \qquad x\leq 0 \tag{4.5}$$

$$Q_{\gamma,n}(o,t) = 0 \qquad t\geq 0 \tag{4.6}$$

[redacted] The uniqueness of such functions is well known and follows, for example, from the results of Widder.[7] The continuity of $Q_{\gamma,n}$ for $t\geq 0$ is established in Proposition (1.3.1) of Reference 1. (The restriction $\gamma+n>-2$ is required here.) The homogeneous initial and boundary values follow directly from substitution of (3.15) and (3.16) into (4.1). The boundedness requirement

[7] *D. V. Widder, "The Heat Equation", p-139, Academic Press, New York, 1975.*

is directly verified by applying inequality (3.23) to the individual terms of $Q_{\gamma,n}$. Thus it only remains to show that $Q_{\gamma,n}$ satisfies (4.4). From (3.20) we have

$$[D_t - D_x^2]Q_{\gamma,n} = -\sum_{k=1}^{n+1}(-\tfrac{1}{2})^{n+2-k}\,[2H_{\gamma+n+1-k,k-1} + H_{\gamma+n+2-k,\,k-2}]$$

$$= \sum_{k=1}^{n+1}(-\tfrac{1}{2})^{n+1-k}\,H_{\gamma+n+1-k,\,k-1} - \sum_{k=2}^{n+1}(-\tfrac{1}{2})^{n+2-k}\,H_{\gamma+n+2-k,k-2}$$

where the k=1 term in the second sum was omitted in view of (3.14). Shifting the index in this sum by setting $j = k-1$ we have

$$[D_t - D_x^2]Q_{\gamma,n} = \sum_{k=1}^{n+1}(-\tfrac{1}{2})^{n+1-k}\,H_{\gamma+n+1-k,\,k-1}$$

$$- \sum_{j=1}^{n}(-\tfrac{1}{2})^{n+1-j}\,H_{\gamma+n+1-j,\,j-1}$$

$$= H_{\gamma,n}$$

This completes the proof.

PROPOSITION (4.2): If $\gamma+n > -2$ then (4.2) defines the unique function which is continuous for $t>0$, satisfies the equation

$$[D_t - D_x^2]\;P_{\gamma,n}(x,t) = H_{\gamma,n}(x,t) \qquad (4.7)$$

in H, with initial values

$$P_{\gamma,n}(x,o) = 0 \qquad x\in R \qquad (4.8)$$

and which is bounded such that for any $T>0$ constants k, $K\geq 0$ exist for which

$$|P_{\gamma,n}(x,t)| \leq K\exp(kx^2)$$

uniformly for $o\leq t\leq T$.

PROOF: Essentially the same as for Proposition (4.1) except that (4.8) must be established for $x\geq 0$. But the constant $c_{\gamma,n}$ was chosen precisely so that this result holds, as may be seen by substituting (3.15) into (4.2) with t=0.

REMARK (4.1): If we denote by $\mathcal{H}_\lambda$ the family of all finite linear combinations of functions $H_{\gamma,n}$ with $\gamma+n=\lambda$ then clearly $P_{\gamma,n}$ and $Q_{\gamma,n}$ belong to $\mathcal{H}_{\gamma+n}$. The importance of the above

propositions is that for any v in $\mathcal{H}_\lambda$ the solution to

$$[D_t - D_x^2]\, u = v$$

with homogeneous initial/boundary data can be found in the class $\mathcal{H}_{\lambda+2}$. It is this behavior which enables us to obtain solutions for the equations of Section II explicitly in terms of the $H_{\gamma,n}$ functions. In fact, for the canonical problems $(\mathrm{IVP})_0$ and $(\mathrm{BVP})_0$, the term u_k of the asymptotic expansion (2.9) will be found in the class $\mathcal{H}_{\gamma+k}$.

REMARK [illegible] For any $a > 0$ the foregoing results imply that the functions $P_{\gamma,n}(x,at)$ and $Q_{\gamma,n}(x,at)$ solve the equation

$$[D_t - aD_x^2]\, u(x,t) = aH_{\gamma,n}(x,at)$$

in place of (4.4) and (4.7) and satisfy the same homogeneous initial/boundary data as before.

REMARK (4. [illegible]) The functions

$$P^*_{\gamma,n}(x,t) = P_{\gamma,n}(-x,t)$$

and

$$Q^*_{\gamma,n}(x,t) = Q_{\gamma,n}(-x,t)$$

satisfy the equations

$$[D_t - D_x^2]\, P^*_{\gamma,n} = H^*_{\gamma,n}$$

in H, and

$$[D_t - D_x^2]\, Q^*_{\gamma,n} = H^*_{\gamma,n}$$

in Q, and take on the initial and boundary values

$$P^*_{\gamma,n}(x,o) = 0 \qquad x \in R$$

$$Q_{\gamma,n}(x,o) = 0 \qquad x \geq 0$$

$$Q_{\gamma,n}(o,t) = 0 \qquad t \geq 0.$$

V. DIFFUSION EQUATION SOLUTION SEQUENCES: THE FUNCTIONS $E_{\gamma,n}$ AND $F_{\gamma,n}$

We have now reached the point at which we can formally construct solutions for the equations derived in Section II.

Let $N \geq 0$ be an integer and let Π_N be an (ordered) parameter set given by

$$\Pi_N = \{a_0, a_1, \ldots, a_N, b_0, b_1, \ldots, b_{N-1}, c_0, c_1, \ldots, c_{N-2}\}.$$

These parameters may of course be considered as the Taylor coefficients of $a(x)$, $b(x)$ and $c(x)$ at $x=x_o$ but their genesis is unimportant here and the discussion may proceed independently of subsequent applications.

For $n=0,1, \ldots, N$ we may define operators L_n by

$$L_n u = a_n \frac{x^n}{n!} D_x^2 u + b_{n-1} \frac{x^{n-1}}{(n-1)!} D_x u + c_{n-2} \frac{x^{n-2}}{(n-2)!} u. \quad (5.1)$$

In particular

$$L_o u = a_o D_x^2 u$$

$$L_1 u = a_1 x D_x^2 u + b_o D_x u.$$

A sequence of functions $U = \{u_n(x,t): n = 0,1,\ldots, N\}$ defined in the closure of a domain $D \subset H$ will be called a <u>Diffusion Equation Solution Sequence</u> (DESS) in D with respect to the parameter set Π_N if the following equations are satisfied

$$[D_t - L_o]u_n = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{k=1}^{n} L_k u_{n-k} & \text{if } n = 1,2, \ldots, N \end{cases} \quad (5.2)$$

If D is unbounded we also require that for any $T>0$ there exist constants k, $K \geq 0$ such that

$$|u_n(x,t)| \leq K \exp (k x^2) \quad (5.3)$$

Uniformly in $0 \leq t \leq T$ for each $n = 0,1, \ldots, N$. We shall use the notation $U \in \mathcal{D}(D, \Pi_N)$ or, more simply, $U \in \mathcal{D}_N$ to indicate that a sequence $U = \{u_n: n = 0,1, \ldots, N\}$ is of the type described.

If $U,V \in \mathcal{D}_N$ and $\alpha, \beta \in R$ then we may define the formal sum sequence

$$W = \alpha U + \beta V$$

termwise by

$$w_n = \alpha u_n + \beta v_n$$

and, for $0 \leq k \leq N$, we may define the "shifted" sequence $U^{(k)}$ termwise as the sequence whose n-th term is

$$u_n^{(k)} = \begin{cases} 0 & 0 \leq n \leq k-1 \\ u_{n-k} & k \leq n \leq N. \end{cases}$$

(The superscript k here is not meant to imply differentiation.)

It should be clear that these sequences are also of class $\mathcal{D}_N$. Moreover, let $d_k \in R$ and $U_k \in \mathcal{D}_N$ for $k = 0,1, \ldots, N$, with $U_k = \{U_{k,n}\}$, and define the sequence of "shifted" partial sums by

$$V = \sum_{k=0}^{N} d_k U_k^{(k)}$$

or termwise by

$$v_n = \sum_{k=0}^{n} d_k u_{k,n-k} = \sum_{k=0}^{n} d_k u_{k,n}^{(k)}$$

Then the previous observation implies that $V \in \mathcal{D}_N$.

From the existence and uniqueness properties of the diffusion equation, $D_t u = L_o u$, we see that we may specify the individual terms of a DESS by requiring that they also satisfy certain boundary and initial data. The following sequences will be of particular interest:

(1) $E_{\gamma,n}$ and $E\#_{\gamma,n}$ defined in H and satisfying

$$E_{\gamma,n}(x,0) = \begin{cases} h_\gamma(x) & \text{if } n = 0 \\ 0 & \text{if } n = 1,2,\ldots,N. \end{cases} \tag{5.4}$$

$$E\#_{\gamma,n}(x,0) = \begin{cases} h^*_\gamma(x) & \text{if } n = 0 \\ 0 & \text{if } n = 1,2,\ldots, N. \end{cases} \tag{5.4\#}$$

(2) $F_{\gamma,n}$ and $F^{\#}_{\gamma,n}$ defined in $\overline{Q}^*$ and $\overline{Q}$ respectively and satisfying

$$F_{\gamma,n}(x,0) = 0 \text{ for } x \le 0, \; n = 0,1,\ldots,N \tag{5.5}$$

$$F\#_{\gamma,n}(x,0) = 0 \text{ for } x \ge 0, \; n = 0,1,\ldots,N \tag{5.5\#}$$

$$F_{\gamma,n}(0,t) = F\#_{\gamma,n}(0,t) = \begin{cases} \dfrac{\sqrt{t}^{\,\gamma}}{(\gamma/2)!} & n = 0 \\ 0 & n = 1,2,\ldots,N \end{cases} \tag{5.6}$$

for $t > 0$.

The following two propositions are rather technical but establish that each of the terms of the foregoing DESS's is expressible as a linear combination of functions $H_{\gamma,n}$ and $H^*_{\gamma,n}$ with coefficients that can be explicitly determined from recursion formulae. The boundedness condition (5.3) will then be automatically satisfied due to the growth properties of such functions [see (3.23)].

[redacted] Let $0 \le n \le N$, then

$$E_{\gamma,n}(x,t) = \sum_{k=0}^{2n} \alpha_{\gamma,n,k} H_{\gamma+n-k,\,k}(x, a_o t) \tag{5.7}$$

where the coefficients $\alpha_{\gamma,n,k}$ are determined by the following recursive formulae:

$$\text{for } n = 0 \qquad \alpha_{\gamma,0,0} = 1$$

$$\text{for } n = 1,2,\ldots,N \quad \alpha_{\gamma,n,k} = \frac{1}{a_o} \begin{cases} -\displaystyle\sum_{j=0}^{2n-1} c_{\gamma+n-2-j,\,j}\,\bar{\alpha}_{\gamma,n,j} & \text{if } k = 0 \\ \displaystyle\sum_{j=1}^{2n-k+1} \left(\frac{-1}{2}\right)^{j} \bar{\alpha}_{\gamma,n,j+k-2} & \text{if } k = 1,2,\ldots,2n \end{cases} \tag{5.8}$$

where, from (4.3)

$$c_{\gamma+n-2-j,\,j} = \sum_{i=1}^{j+1} \left(\frac{-1}{2}\right)^{j+2-i} \binom{\gamma+n}{i}$$

and the $\bar{\alpha}_{\gamma,n,j}$ are defined for $j = 0,1,\ldots,2n-1$ by

$$\begin{aligned} \bar{\alpha}_{\gamma,n,j} = {} & \sum_{i=1}^{I} a_i \binom{j}{i} \alpha_{\gamma,n-i,j-i} \\ & \cdot \sum_{i=1}^{J} \left[2a_i \binom{j}{i} + b_{i-1} \binom{j}{i-1} \right] \alpha_{\gamma,n-i,j-i+1} \\ & \cdot \sum_{i=1}^{K} \left[a_i \binom{j}{i} + b_{i-1} \binom{j}{i-1} + c_{i-2} \binom{j}{i-2} \right] \alpha_{\gamma,n-i,j+2-i} \end{aligned} \tag{5.9}$$

with the upper summation limits being given by

$$\begin{aligned} I &= \begin{cases} j & \text{if } 0 \le j \le n \\ 2n-j & \text{if } n+1 \le j \le 2n-1 \end{cases} \\ J &= \begin{cases} j+1 & \text{if } 0 \le j \le n-1 \\ 2n-j-1 & \text{if } n \le j \le 2n-1 \end{cases} \\ K &= \begin{cases} j+2 & \text{if } 0 \le j \le n-2 \\ 2n-j-2 & \text{if } n-1 \le j \le 2n-1 \end{cases} \end{aligned} \tag{5.10}$$

It should be understood that the sums appearing in (5.9) are omitted when the lower summation limit exceeds the upper.

Proof. The proposition holds for $n = 0$ since we then have

$$E_{\gamma,o}(x,t) = H_{\gamma,o}(x,a_o t)$$

Proceeding by induction we may assume that (5.7) is valid for $n = 0,1,\ldots,m-1$ where $1 \leq n \leq N$ and show that is must also hold for $n = m$. For convenience in this proof we shall write $H_{\gamma,n}$ for $H_{\gamma,n}(x,a\ t)$. The k-th term on the right hand side of (5.2), when $u_n = E_{\gamma,n}$ and $n = m$ is then

$$L_k E_{\gamma,m-k} = a_k \frac{x^k}{k!} \sum_{j=0}^{2(m-k)} \alpha_{\gamma,m-k,j} D_x^2 H_{\gamma+m-k-j,j}$$

$$+ b_{k-1} \frac{x^{k-1}}{(k-1)!} \sum_{j=0}^{2(m-k)} \alpha_{\gamma,m-k,j} D_x H_{\gamma+m-k-j,j}$$

$$+ c_{k-2} \frac{x^{k-2}}{(k-2)!} \sum_{j=0}^{2(m-k)} \alpha_{\gamma,m-k,j} H_{\gamma+m-k-j,j}$$

where $1 \leq k \leq m$. Using (3.17) and (3.18) yields

$$L_k E_{\gamma,m-k} = a_k \frac{x^k}{k!} \sum_{j=0}^{2(m-k)} \alpha_{\gamma,m-k,j}[H_{\gamma+m-2-k-j,j}$$

$$+2H_{\gamma+m-1-k-j,j-1} + H_{\gamma+m-k-j,j-2}]$$

$$+b_{k-1} \frac{x^{k-1}}{(k-1)!} \sum_{j=0}^{2(m-k)} \alpha_{\gamma,m-k,j}[H_{\gamma+m-1-k-j,j} + H_{\gamma+m-k-j,j-1}]$$

$$+c_{k-2} \frac{x^{k-2}}{(k-2)!} \sum_{j=0}^{2(m-k)} \alpha_{\gamma,m-k,j} H_{\gamma+m-k-j,j}.$$

Using the identity

$$\frac{x^m}{m!} H_{\gamma,n}(x,t) = \binom{m+n}{n} H_{\gamma,m+n}(x,t)$$

which follows directly from definition (3.12) we then have

$$L_k E_{\gamma,m-k} = \sum_{j=o}^{2(m-k)} a_k \binom{j+k}{k} \alpha_{\gamma,m-k,j} H_{\gamma+m-2-j-k,j+k}$$

$$+\sum_{j=0}^{2(m-k)} [2a_k \binom{j+k-1}{k} + b_{k-1} \binom{j+k-1}{k-1}] \alpha_{\gamma,m-k,j}$$

$$\cdot H_{\gamma+m-1-j-k,j+k-1}$$

$$+\sum_{j=0}^{2(m-k)} [a_k \binom{j+k-2}{k} + b_{k-1} \binom{j+k-2}{k-1} + c_{k-2} \binom{j+k-2}{k-2}]$$

$$\cdot \alpha_{\gamma,m-k,j} H_{\gamma+m-j-k,j+k-2}$$

The right hand side of (5.2) when $u_n = E_{\gamma,n}$ and $n = m$ is obtained by summing this expression with respect to $k = 1,2,\ldots,m$. For $m = 1$ this reduces to

$$L_1 E_{\gamma,o} = a_1 H_{\gamma-2,1} + b_o H_{\gamma-1,o}$$

For $m \geq 2$ we may reexpress the summation with the help of identity (1.4); thus

$$\sum_{k=1}^{m} L_k E_{\gamma,m-k} = \sum_{n=1}^{m} \sum_{i=1}^{n} a_i \binom{n}{i} \alpha_{\gamma,m-i,n-i} H_{\gamma+m-2-n,n}$$

$$+ \sum_{n=m+1}^{2m-1} \sum_{i=1}^{2m-n} a_i \binom{n}{i} \alpha_{\gamma,m-i,n-i} H_{\gamma+m-2-n,n}$$

$$+ \sum_{n=1}^{m} \sum_{i=1}^{n} [2a_i \binom{n-1}{i} + b_{i-1} \binom{n-1}{i-1}] \alpha_{\gamma,m-i,n-i}$$

$$\cdot H_{\gamma+m-1-n,n-1}$$

$$+ \sum_{n=m+1}^{2m-1} \sum_{i=1}^{2m-n} [2a_i \binom{n-1}{i} + b_{i-1} \binom{n-1}{i-1}] \alpha_{\gamma,m-i,n-i}$$

$$\cdot H_{\gamma+m-1-n,n-1}$$

$$+ \sum_{n=1}^{m} \sum_{i=1}^{n} [a_i \binom{n-2}{i} + b_{i-1} \binom{n-2}{i-1} + c_{i-2} \binom{n-2}{i-2}]$$

$$\cdot \alpha_{\gamma,m-i,n-i} H_{\gamma+m-n,n-2}$$

$$+\sum_{n=m+1}^{2m-1}\sum_{i=1}^{2m-n} a_i\left[\binom{n-2}{i}+b_{i-1}\binom{n-2}{i-1}+c_{i-2}\binom{n-2}{i-2}\right]$$

$$\cdot\alpha_{\gamma,m-i,n-i}H_{\gamma+m-n,n-2}. \tag{5.11}$$

Shifting indices and rearranging slightly we get

$$\sum_{k=1}^{m} L_k E_{\gamma,m-k} = \sum_{n=1}^{m}\sum_{i=1}^{n} a_i\binom{n}{i}\alpha_{\gamma,m-i,n-i}H_{\gamma+m-2-n,n}$$

$$+\sum_{n=m+1}^{2m-1}\sum_{i=1}^{2m-n} a_i\binom{n}{i}\alpha_{\gamma,m-i,n-i}H_{\gamma+m-2-n,n} \tag{5.12}$$

$$+\sum_{n=0}^{m-1}\sum_{i=1}^{n+1}\left[2a_i\binom{n}{i}+b_{i-1}\binom{n}{i-1}\right]\alpha_{\gamma,m-i,n+1-i}H_{\gamma+m-2-n,n}$$

$$+\sum_{n=m}^{2m-2}\sum_{i=1}^{2m-n-1}\left[2a_i\binom{n}{i}+b_{i-1}\binom{n}{i-1}\right]\alpha_{\gamma,m-i,n+1-i}H_{\gamma+m-2-n,n}$$

$$+\sum_{n=0}^{m-2}\sum_{i=1}^{n+2}\left[a_i\binom{n}{i}+b_{i-1}\binom{n}{i-1}+c_{i-2}\binom{n}{i-2}\right]\alpha_{\gamma,m-i,n+2-i}H_{\gamma+m-2-n,n}$$

$$+\sum_{n=m-1}^{2m-3}\sum_{i=1}^{2m-n-2}\left[a_i\binom{n}{i}+b_{i-1}\binom{n}{i-1}+c_{i-2}\binom{n}{i-2}\right]\alpha_{\gamma,m-i,n+2-i}H_{\gamma+m-2-n,n}$$

Collecting together the coefficients of similar terms we can then write

$$\sum_{k=1}^{m} L_k E_{\gamma,m-k} = \sum_{n=0}^{2m-1}\bar{\alpha}_{\gamma,m,n}H_{\gamma+m-2-n,n} \tag{5.13}$$

where the $\bar{\alpha}_{\gamma,m,n}$ are given by (5.9) above for $m \geq 1$. In the case $m=1$ most of the terms are absent leaving only

$$\bar{\alpha}_{\gamma,1,0} = b_o \quad\text{and}\quad \bar{\alpha}_{\gamma,1,1} = a_1.$$

Equation (5.2) applied to the m-th term of the DESS $\{E_{\gamma,n}\}$ can thus be written as

$$[D_t - a_o D_x^2]E_{\gamma,m}(x,t) = \sum_{n=0}^{2m-1}\bar{\alpha}_{\gamma,m,n}H_{\gamma+m-2-n,n}(x,a_o t).$$

Recalling Proposition (4.2) and Remark (4.2) and using linearity it immediately follows that for m = 1,2...

$$E_{\gamma,m}(x,t) = \frac{1}{a_o} \sum_{n=0}^{2m-1} \bar{\alpha}_{\gamma,m,n} P_{\gamma+m-2-n,n}(x,a_o t)$$

$$= \frac{1}{a_o} \sum_{n=0}^{2m-1} \bar{\alpha}_{\gamma,m,n} \Big[\sum_{k=1}^{n+1} \left(\frac{-1}{2}\right)^{n+2-k} H_{\gamma+m-k,k}(x,a_o t)$$

$$- c_{\gamma+m-2-n,n} H_{\gamma+m}(x,a_o t) \Big]$$

By collecting coefficients with the help of (1.3) this can be rewritten as

$$E_{\gamma,m}(x,t) = \sum_{k=0}^{2m} \alpha_{\gamma,m,k} H_{\gamma+m-k,k}(x,a_o t)$$

where $\alpha_{\gamma,m,k}$ is given by (5.8) with $n = m \geq 1$. This completes the proof.

For $n = 0,1,2$ the coefficients $\alpha_{\gamma,n,k}$ are listed explicitly in the Appendix. For convenience let us also define

$$\alpha_{\gamma,n,k} = 0 \qquad (5.14)$$

for $k < 0$ or $k > 2n$.

PROPOSITION (5.2): Let $0 \leq n \leq N$. Then

$$F_{\gamma,n} = \frac{2}{\sqrt{a_o}^{\gamma}} \sum_{k=0}^{2n} \beta_{\gamma,n,k} H_{\gamma+n-k,k}(x,a_o t) \qquad (5.15)$$

where the coefficients $\beta_{\gamma,n,k}$ are defined by the following recursion formulae

for $n = 0$ $\qquad \beta_{\gamma,0,0} = 1$

$$\text{for } n = 1,2,\ldots,N \quad \beta_{\gamma,n,k} = \begin{cases} 0 & \text{if } k = 0 \\ \dfrac{1}{a_o} \displaystyle\sum_{j=1}^{2n-k+1} \left(\frac{-1}{2}\right)^{j} \bar{\beta}_{\gamma,n,j+k-2} & \text{if } k = 1,2,\ldots,2n \end{cases} \qquad (5.16)$$

with $\bar{\beta}_{\gamma,n,k}$ being defined for $k = 0,1,\ldots,2n-1$ by

$$\bar{\beta}_{\gamma,n,k} = \sum_{i=1}^{I} a_i \binom{k}{i} \beta_{\gamma,n-i,k-i}$$

$$+\sum_{i=1}^{J}\ [2a_i \binom{k}{i} + b_{i-1}\binom{k}{i-1}]\ \beta_{\gamma,n-i,k+1-i} \qquad (5.17)$$

$$+\sum_{i=1}^{K}\ [a_i \binom{k}{i} + b_{i-1}\binom{k}{i-1} + c_{i-2}\binom{k}{i-2}]\ \beta_{\gamma,n-i,k+2-i}$$

where the upper summation limits I, J and K are given by (5.10).

PROOF: The proof is identical to that of Proposition (5.1) up to the point where in place of (5.13) we would obtain

$$\sum_{k=1}^{m} L_k F_{\gamma,m-k} = \frac{2}{\sqrt{a_o}^{\gamma}} \sum_{n=0}^{2m-1} \bar{\beta}_{\gamma,m,n} H_{\gamma+m-2-n,n}$$

for $m > 1$ where the coefficients $\bar{\beta}_{\gamma,m,n}$ are given by (5.17). In particular for $m = 1$

$$\bar{\beta}_{\gamma,1,0} = b_o$$
$$\bar{\beta}_{\gamma,1,1} = a_1.$$

Because of the different boundary and initial data in the present case however we now use Proposition (4.1) instead of Proposition (4.2) *and obtain*

$$F_{\gamma,m}(x,t) = \frac{2}{\sqrt{a_o}^{\gamma}} \sum_{n=0}^{2m-1} \beta_{\gamma,m,n} \frac{1}{a_o} Q_{\gamma+m-2-n,n}(x,a_o t)$$

$$= \frac{2}{a_o} \sum_{n=0}^{2m-1} \beta_{\gamma,m,n} \frac{1}{a_o} \sum_{k=1}^{n+1} \left(\frac{-1}{2}\right)^{n+2-k} H_{\gamma+m-k,k}(x,a_o t)$$

which can be rewritten as

$$F_{\gamma,n}(x,t) = \frac{2}{\sqrt{a_o}^{\gamma}} \sum_{k=1}^{2m} \beta_{\gamma,m,k} H_{\gamma+m-k,k}(x,a_o t)$$

where the $\beta_{\gamma,m,n}$ are given by (5.16) above for $m = 1,2,\ldots,N$. This completes the proof.

For $n = 0,1,2$ the coefficients $\beta_{\gamma,n,k}$ are given explicitly in the Appendix. As with the $\alpha_{\gamma,n,k}$ let use define for convenience

$$\beta_{\gamma,n,k} = 0 \qquad (5.18)$$

for $k < 0$ and $k > 2n$.

The following lemma permits us to characterize the functions $E^{\#}_{\gamma,k}$ and $F^{\#}_{\gamma,k}$ directly in terms of the functions $E_{\gamma,k}$ and $F_{\gamma,k}$.

LEMMA [illegible].1: If $\{u_n : n = 0,1,\ldots,N\} \in \mathcal{D}_N(D,\Pi_N)$ and $u_n^{\#}$ is defined by

$$u_n^{\#}(x,t) = (-1)^n u_n^*(x,t)$$

then $\{u_n^{\#} : n = 0,1,\ldots,N\} \in \mathcal{D}_N(D^*,\Pi_N)$; here $u_n^*(x,t) \equiv u_n(-x,t)$.

PROOF: Let us transform (5.2) by $x = -y$ and $D_x = -D_y$. Then we have

$$[D_t - a_o D_y^2] u_n(-y,t) = \sum_{k=1}^{n} [a_k \frac{(-y)^k}{k!} D_y^2 - b_{k-1} \frac{(-y)^{k-1}}{(k-1)!} D_y$$

$$+ c_{k-2} \frac{(-y)^{k-2}}{(k-2)!}] u_n(-y,t)$$

$$= \sum_{k=1}^{n} (-1)^k [a_k \frac{y^k}{k!} D_y^2 + b_{k-1} \frac{y^{k-1}}{(k-1)!} D_y$$

$$+ c_{k-2} \frac{y^{k-2}}{(k-2)!}] u_n(-y,t).$$

This implies that

$$[D_t - L_o] u_n^* = \sum_{k=1}^{n} (-1)^k L_k u_{n-k}^*.$$

Multiplying across by $(-1)^n$ we then obtain

$$[D_t - L_o] u_n^{\#} = \sum_{k=1}^{n} L_k u_{n-k}^{\#}$$

which concludes the proof.

Clearly then we may set

$$E^{\#}_{\gamma,n} = (-1)^n E^*_{\gamma,n} \tag{5.19}$$

and

$$F^{\#}_{\gamma,n} = (-1)^n F^*_{\gamma,n} \quad n = 0,1,\ldots,N \tag{5.20}$$

since these functions will satisfy the correct initial and boundary conditions.

[redacted] The coefficients $\alpha_{\gamma,n,k}$ and $\beta_{\gamma,n,k}$ have been listed explicitly in Appendix B for $n = 0,1,2$ because they quickly become tedious to calculate as n increases. Perhaps symbol manipulation programs, which have come into vogue, would be quite useful in this regard.

[redacted] Although $F_{\gamma,n}$ and $F^{\#}_{\gamma,n}$ were originally defined only in the quarter planes $\bar{Q}$ and $\bar{Q}^*$ their representations (5.15) and (5.20) are defined throughout $\bar{H}$. This permits these functions to be analytically continued (with respect to x) into all of $\bar{H}$ in such a manner that (5.2)$_{\#}$ remains valid in the extended domain H. Thus $\{F_{\gamma,n}\}$ and $\{F^{\#}_{\gamma,n}\}$ are of class $\mathcal{D}_N(H,\Pi_N)$.

[redacted] By setting $x = 0$ in (5.7) and using (3.16) and (3.14) we have

$$E_{\gamma,n}(0,t) = \frac{1}{2}\alpha_{\gamma,n,o}\frac{\sqrt{a_o t}^{\,\gamma+n}}{((\gamma+n)/2)!} \tag{5.21}$$

and then from (5.19)

$$E^{\#}_{\gamma,n}(0,t) = \frac{(-1)^n}{2}\alpha_{\gamma,n,o}\frac{\sqrt{a_o t}^{\,\gamma+n}}{((\gamma+n)/2)!} \tag{5.21*}$$

By setting $t = 0$ in (5.15) we have

$$F_{\gamma,n}(x,0) = \frac{2}{\sqrt{a_o}^{\,\gamma}}\sum_{k=0}^{2n}\binom{\gamma+n}{k}\beta_{\gamma,n,k}\frac{x^{\gamma+n}}{(\gamma+n)!}, \qquad x \geq 0 \tag{5.22}$$

$$F^{\#}_{\gamma,n}(x,0) = \frac{2(-1)^n}{\sqrt{a_o}^{\,\gamma}}\sum_{k=0}^{2n}\binom{\gamma+n}{k}\beta_{\gamma,n,k}\frac{(-x)^{\gamma+n}}{(\gamma+n)!}, \qquad x \leq 0 \tag{5.22*}$$

[redacted] For any $a > 0$, substituting (3.22) into (5.7) and (5.15) easily leads to the following identities

$$E_{\gamma,k}(x,t) = \sqrt{a}^{\,\gamma+k}\, E_{\gamma,k}(x/\sqrt{a},\ t/a) \tag{5.23}$$

$$F_{\gamma,k}(x,t) = \sqrt{a}^{\,\gamma+k}\, F_{\gamma,k}(x/\sqrt{a},\ t/a) \tag{5.24}$$

VI. FORMAL EXPANSION PROCEDURES--CONTINUED

At the end of Section II we obtained a system of equations and initial/boundary data for the individual terms $\tilde{u}_k$ in the assumed form (2.9) *of an asymptotic expansion* for $u(x,t)$. Comparing equations (5.2) and (5.4) - (5.6) which defined $E_{\gamma,k}$ and $F_{\gamma,k}$ in Section V with the equations (2.11) and (2.13) - (2.15) formulated for $\tilde{u}_k$ we immediately obtain the solutions

$$\tilde{u}_k(\sigma,\tau) = E_{\gamma,k}(\sigma,\tau)$$

for problem $(IVP)_o$ and

$$\tilde{u}_k(\sigma,\tau) = F_{\gamma,k}(\sigma,\tau)$$

for problem $(BVP)_o$ where $E_{\gamma,k}$ and $F_{\gamma,k}$ are defined by (5.7) and (5.15) respectively. Substitution back into the formal asymptotic expansion then yields

$$\tilde{u}(\sigma,\tau) \sim \varepsilon^{\gamma} \sum_{k=0}^{\infty} \varepsilon^{k} E_{\gamma,k}(\sigma,\tau)$$

and

$$\tilde{u}(\sigma,\tau) \sim \varepsilon^{\gamma} \sum_{k=0}^{\infty} \varepsilon^{k} F_{\gamma,k}(\sigma,\tau).$$

By re-expressing these expansions in terms of the original variables x and t, related to σ and τ by (2.7), and truncating to N terms we obtain the following functions

$$V^{N}_{\gamma,x_o}(x,t) \equiv \sum_{k=0}^{N} \varepsilon^{\gamma+k} E_{\gamma,k}((x-x_o)/\varepsilon,\ t/\varepsilon^2)$$

$$W^{N}_{\gamma,x_o}(x,t) \equiv \sum_{k=0}^{N} \varepsilon^{k} F_{\gamma,k}((x-x_o)/\varepsilon,\ t/\varepsilon^2)$$

Recalling identities (5.23), and (5.24) these can be written more simply as

$$V^{N}_{\gamma,x_o}(x,t) = \sum_{k=0}^{N} E_{\gamma,k}(x-x_o,t) \tag{6.1}$$

$$W^N_{\gamma,x_o}(x,t) = \sqrt{\Delta t}^{\,-\gamma} \sum_{k=0}^{N} F_{\gamma,k}(x-x_o,t) \tag{6.2}$$

or, in the more expanded form,

$$V^N_{\gamma,x_o}(x,t) = \sum_{k=0}^{N} \sum_{j=0}^{2k} \alpha_{\gamma,k,j} \; H_{\gamma+k-j,j}\,(x-x_o,a_o t) \tag{6.3}$$

$$W^N_{\gamma,x_o}(x,t) = 2\sqrt{a_o \Delta t}^{\,-\gamma} \sum_{k=0}^{N} \sum_{j=0}^{2k} \beta_{\gamma,k,j} H_{\gamma+k-j,j}(x-x_o,a_o t) \tag{6.4}$$

For k = 0,1,2 the coefficients $\alpha_{\gamma,k,j}$ and $\beta_{\gamma,k,j}$ are given explicitly in the Appendix; more generally they can be determined using the recursion formulae of Propositions (5.1) and (5.2). Consideration of the converse problems $(IVP)^{\#}_{o}$ and $(BVP)^{\#}_{o}$ lead to the analogous expansions

$$V^{\#N}_{\gamma,x_o}(x,t) = \sum_{k=0}^{N} E^{\#}_{\gamma,k}\;(x-x_o,t) \tag{6.5}$$

and

$$W^{\#N}_{\gamma,x_o}(x,t) = \sqrt{\Delta t}^{\,\gamma} \sum_{k=0}^{N} F^{\#}_{\gamma,k}(x-x_o,t) \tag{6.6}$$

which can also be written, using (5.19) and (5.20), as

$$V^{\#N}_{\gamma,x_o}(x,t) = \sum_{k=0}^{N} (-1)^k \sum_{j=0}^{2k} \alpha_{\gamma,k,j} H^{*}_{\gamma+k-j,j}(x-x_o,a_o t) \tag{6.7}$$

$$W^{\#N}_{\gamma,x_o}(x,t) = 2\sqrt{a_o \Delta t}^{\,-\gamma} \sum_{k=0}^{N} (-1)^k \sum_{j=0}^{2k} \beta_{\gamma,k,j} H^{*}_{\gamma+k-j,j}(x-x_o,a_o t) \tag{6.8}$$

The expansions V^N_{γ,x_o} and $V^{\#N}_{\gamma,x_o}$ will be referred to as the <u>N-term special expansions at x_o with respect to the operator L</u> or, more briefly, the special expansions. The expansions

$$W^N_{\gamma,x_o} \quad \text{and} \quad W^{\#N}_{\gamma,x_o}$$

will be termed the <u>N-term right and left hand boundary expansions</u> (respectively) <u>at x_o with respect to the operator L.</u>

The values taken on by the special expansions at $x=x_o$ are of special importance and can be obtained by substituting (3.16) into (6.3) and (6.7) with $x = x_o$. We have

$$V^N_{\gamma,x_o}(x_o,t) = \frac{1}{2} \sum_{k=0}^{N} \alpha_{\gamma,k,o} \frac{\sqrt{a_o t}^{\,\gamma+k}}{((\gamma+k)/2)!} \tag{6.9}$$

and

$$V^{\#}_{\gamma,x_o}(x_o,t) = \frac{1}{2} \sum_{k=0}^{N} (-1)^k \alpha_{\gamma,k,o} \frac{\sqrt{a_o t}^{\,\gamma+k}}{((\gamma+k)/2)!} \tag{6.9\#}$$

In the next two sections we shall develop uniformly accurate approximations for the solution of the Cauchy problem and the first initial-boundary value problem for equation (2.1) using the expansions

$$V^N_{\gamma,x_o}, \quad W^N_{\gamma,x_o}, \quad V^{\#N}_{\gamma,x_o} \text{ and } W^{\#N}_{\gamma,x_o}.$$

Note that these functions are well defined whenever the Taylor coefficients

$$\begin{array}{ll} a_k & k = 0,1,\dots,N \\ b_k & k = 0,1,\dots,N-1 \\ c_k & k = 0,1,\dots,N-2 \end{array}$$

exist at $x = x_0$. In order to establish rigorous error bounds for the approximations of the next two sections, however, it will be necessary to make slightly stronger assumptions in the form

$$\text{(A)} \qquad \begin{array}{lll} a(x) \in C^{N+\alpha}(I) & & \\ b(x) \in C^{N+1+\alpha}(I) & & (\text{if } N \geq 1) \\ c(x) \in C^{N+2+\alpha}(I) & & (\text{if } N \geq 2) \end{array}$$

where $N \geq 0$ is an integer, $0 \leq \alpha \leq 1$ and I is an interval with $x_0 \in I$; if x_0 corresponds to a boundary of the solution domain for (2.1) then x_o is taken as an end point of I, otherwise x_o must be an interior point of I.

VII. THE CAUCHY PROBLEM

The Cauchy problem is formulated by equations (2.1) and (2.2)

$$D_t u = L u \qquad x \in R, \quad t > 0$$

$$u(x,o) = f(x) \qquad x \in R$$

If a, b and c are bounded and Holder continuous and f is continuous then this problem is known[3] to have a unique solution in the class of functions satisfying (2.6). If, in addition, a, b and c are sufficiently differentiable then the expansions developed in the previous sections can be used to approximate u(x,t) with uniform accuracy. A rigorous analysis of this problem was undertaken in Reference 1 and in this section we wish to present the main results in a convenient, abbreviated form without proofs.

If f(x) is sufficiently smooth then only the regular expansion (2.3) is needed to approximate u(x,t), as we see in the following two theorems. The first is a global result requiring stronger hypotheses than the second which is a local result. In the following we shall use the notation

$$\overline{N} = [N/2] = \text{greatest integer} \leq N/2$$

[redacted] Let a, b, c $\in C^{N+\alpha}(R)$ and f $\in C^{N+2+\alpha}(R)$. If $L^{(\overline{N}+1)} f(x)$ is bounded for all $x \in R$ then there exists a constant $K \geq 0$ and a $\Delta t > 0$ such that

$$| u(x,t) - U^{\overline{N}}(x,t) | \leq K \quad t^{\overline{N}+1} \tag{7.1}$$

uniformly in $R \times [o,\Delta t]$.

[redacted] See Theorem (2.2.1) of Reference 1.

[redacted] Let $I \subset R$ be an open interval, let (A) hold in I with $N \geq 2$, $\alpha > 0$ and let $f \in C^{N+\alpha}(I)$. Then there exists a $\Delta t > 0$ and for each compact set $S \subset I$ there exists a constant $K \geq 0$ such that

$$| u(x,t) - U^{\overline{N}}(x,t) | \leq K \sqrt{t}^{N+\alpha} \tag{7.2}$$

uniformly in $S \times [o, \Delta t]$.

[redacted] See Theorem (2.2.4) and Corollary (2.3.1) of Reference 1.

In many practical problems the initial value function is not smooth but can be accurately represented by a piecewise smooth function. Within a bounded interval I this can then be expressed as the sum of a smooth function and (left and right hand) jump functions; that is

$$f(x) = \bar{f}(x) + \sum_{k=1}^{K} \alpha_k h_{\gamma_k}(x - x_k) \tag{7.3}$$

$$+ \sum_{k=1}^{K^*} \alpha^*_k \, h^*_{\gamma^*_k}(x - x^*_k)$$

where $\bar{f}(x)$ is smooth in I, x , $x^* \in$ I and α , $\alpha^* \neq 0$; if no jumps occur then the summation terms are omitted.

Since our problem is linear and since the two previous Theorems apply to $\bar{f}(x)$ then it only remains to treat the two "canonical" cases

$$f(x) = h_\gamma(x - x_o)$$

and

$$f(x) = h^*_\gamma(x - x_o)$$

We shall use the notation

$$u_{\gamma, x_o} \quad \text{and} \quad u^{\#}_{\gamma, x_o}$$

respectively to denote the solutions to the Cauchy problem with these particular choices of initial data. The special expansions (6.1) and (6.5) were constructed in the last four sections for the express purpose of approximating these functions; the following theorem indicates where and how well this is achieved.

THEOREM [illegible]: Let $\gamma \geq 0$, let $I \subset R$ be an open interval with $x_o \in I$ and let (A) hold in I with $N \geq 2$, $\alpha > 0$. Then there exist a constant $K \geq 0$ and a $\Delta t > 0$ such that

$$| u_{\gamma, x_o}(x,t) - V^N_{\gamma, x_o}(x,t) | \leq K \sqrt{t}^{\gamma+N+\alpha} \tag{7.4}$$

uniformly in $(-\infty, x_o] \times [o, \Delta t]$ and

$$| u^{\#}_{\gamma, x_o}(x,t) - V^{\#N}_{\gamma, x_o}(x,t) | \leq K \sqrt{t}^{\gamma+N+\alpha} \tag{7.4\#}$$

uniformly in $[x_o, \infty] \times [o, \Delta t]$.

PROOF: See Theorem (2.3.1) of Reference 1.

This result is not altogether satisfactory since it still does not enable us to write an approximation for

$$u_{\gamma,x_o} \quad \text{or} \quad u^{\#}_{\gamma,x_o}$$

which is uniformly valid in regions such as $S \times [o,\Delta t]$ where, S is a compact set containing x_o as an interior point. However, in the case where γ is a non-negative integer, say $\gamma = n \geq 0$, we can circumvent this limitation using the simple identity

$$\frac{(x-x_o)^n}{n!} = h_n(x-x_o) + (-1)^n \; h^*_n(x-x_o) \tag{7.5}$$

This, together with linearity and uniqueness for the Cauchy problem, implies that

$$\bar{u}_{n,x_o} = u_{n,x_o} + (-1)^n \; u^{\#}_{n,x_o} \tag{7.6}$$

where $\bar{u}_{n,x_o}$ denotes the solution having the polynomial initial values

$$\bar{u}_{n,x_o}(x,o) = (x-x_o)^n/n!$$

Thus, to construct an expansion for u_{n,x_o} valid in the region $x \geq x_o$ we need only re-arrange (7.6) into the form

$$u_{n,x_o} = \bar{u}_{n,x_o} - (-1)^n \; u^{\#}_{n,x_o}.$$

The first term on the right can be approximated using its N-term regular expansion

$$U^N_{n,x_o}(x,t) = \sum_{k=0}^{N} L^{(k)} \; [(x-x_o)^n/n!] \; t^k/k! \tag{7.7}$$

with an error term of order $\sqrt{t}^{N+\alpha}$ (Theorem (7.2)); the second term can be approximated for $x \geq x_o$ using the N-n term special expansion $V^{\#N-n}_{n,x_o}$ with an error of order $\sqrt{t}^{N+\alpha}$ (Theorem (7.3)).

We are thus led to define the following <u>N-th order uniform expansion for u_{n,x_o}</u>

$$Y^N_{n,x_o}(x,t) \equiv \begin{cases} V^{N-n}_{n,x_o}(x,t) & x \leq x_o \\ U^{\bar{N}}_{n,x_o}(x,t) - (-1)^n \; V^{\#N-n}_{n,x_o} & x > x_o \end{cases} \tag{7.8}$$

Similarly we can define the <u>N-th order uniform expansion for $u^{\#}_{n,x_o}$</u> by

$$y^{\#N}_{n,x_o}(x,t) \equiv \begin{cases} (-1)^n \left[U^{\overline{N}}_{n,x_o}(x,t) - V^{N-n}_{n,x_o}(x,t)\right] & x < x_o \\ V^{\#N-n}_{n,x_o}(x,t) & x \geq x_o \end{cases} \tag{7.8\#}$$

We can now state the following result

[redacted] Let (A) hold with $N \geq 2$ in the open interval $I \subset R$ and let $x_o \in I$. Then there exists a $\Delta t > 0$ and for each compact subset $S \subset I$ there exists a constant $K \geq 0$ such that

$$|u_{n,x_o}(x,t) - y^{N}_{n,x_o}(x,t)| \leq K\sqrt{t}^{N+\alpha} \tag{7.9}$$

and

$$|u^{\#}_{n,x_o}(x,t) - y^{\#N}_{n,x_o}(x,t)| \leq K\sqrt{t}^{N+\alpha} \tag{7.9\#}$$

uniformly in $S \times [o, \Delta t]$.

[redacted] Follows directly from the definition of the uniform expansions using Theorem (7.2) and (7.3). This theorem corresponds to Theorem (2.4.1) of Reference 1.

Unfortunately we have not been able to construct an analogous uniformly valid expansions for

$$u_{\gamma,x_o} \quad \text{and} \quad u^{\#}_{\gamma,x_o}$$

in the cases where γ is not an integer. The best that can be done is to note that the regions of validity ($x \leq x_o$ and $x \geq x_o$) for the special expansions can be enlarged slightly to include the parabolically shaped domain $(x-x_o)^2 \leq 4\ Pt$, $0 \leq t \leq \Delta t$ where $P > 0$. This was included in the treatment of the Cauchy problem in Reference 1.

Although the restriction to integer orders does reduce the generality of our theory somewhat it is still sufficiently flexible to treat most physically interesting cases. It does permit the treatment of any initial value function expressed as a piecewise polynomial, for instance. Furthermore, if we limit our consideration to jumps of integer order then, in view of identity (7.5), the general form (7.3) of initial value functions can be rewritten using right (or left) hand jumps only. The following theorem thus represents our most general result for the Cauchy problem.

[redacted] Let $I \subset R$ be an open interval and let (A) hold in I with $N \geq 2$, $\alpha > 0$. Let $u(x,t)$ denote the solution of problem (2.1), (2.2), (2.6) and suppose that $f(x)$ can be written in the form

$$f(x) = \bar{f}(x) + \sum_{k=1}^{K} \sum_{j=0}^{N} \alpha_{kj} h_j (x - x_k)$$

where $\bar{f}(x) \in C^{N+\alpha}(I)$ and $x_k \in I$. Then there exists a $\Delta t > 0$ and for each compact set $S \subset I$ there is a constant $K \geq 0$ such that

$$\left| u(x,t) - \bar{U}^{N}(x,t) - \sum_{k=1}^{K} \sum_{j=0}^{N} \alpha_{kj} Y^{N}_{j,x_k}(x,t) \right| \leq K \sqrt{t}^{N+\alpha} \qquad (7.10)$$

uniformly in $S \times [o, \Delta t]$; here Y_{j,x_k} is defined by (7.8) and $\bar{U}^N$ is the $\bar{N}$-term regular expansion (2.3) for initial values $\bar{f}(x)$:

$$\bar{U}^{N}(x,t) = \sum_{k=0}^{\bar{N}} L^{(k)} \bar{f}(x) \, t^k/k!$$

with $\bar{N} = [N/2]$.

VIII. INITIAL-BOUNDARY VALUE PROBLEMS

Consider the following initial-boundary value problem in the domain $D = (x_1, x_2) \times (o, \Delta t)$

$$\begin{array}{lll} D_t u = Lu & x_o < x < x_1,\ o < t \leq \Delta t & \\ u(x,o) = f(x) & x_o \leq x \leq x_1 & \text{(BVP)} \\ u(x_1,t) = g(t) & o < t \leq \Delta t & \\ u(x_2,t) = h(t) & o < t \leq \Delta t & \end{array}$$

Using linearity this can be broken down as usual into component subproblems by treating each of the functions $f(x)$, $g(t)$ and $h(t)$ in turn, assuming the remaining two function to vanish. We denote the problems specified in this manner by $(BVP)_f$, $(BVP)_g$ and $(BVP)_h$. The latter two problems are nearly identical, of course, and any statement concerning one can be directly transformed into an analogous statement concerning the other. In the following discussion we shall first consider problems $(BVP)_g$ and $(BVP)_h$ and then turn to problem $(BVP)_f$.

The most important particular forms of problems $(BVP)_g$ and $(BVP)_h$ are those with the canonical boundary data

$$g(t) = h_{\gamma/2}(t) \qquad o \leq t \leq \Delta t \tag{8.1}$$

for problem $(BVP)_g$ and

$$h(t) = h_{\gamma/2}(t) \qquad o \leq t \leq \Delta t \tag{8.2}$$

for problem $(BVP)_h$. The boundary expansions

$$W^{\#N}_{\gamma, x_1} \quad \text{and} \quad W^{N}_{\gamma, x_2}$$

were defined by (6.2) and (6.6) with precisely these problems in mind and we have the following result.

THEOREM (8.1). Let γ, N, $\alpha \geq 0$, where N is an integer, and let (A) hold with $N + \alpha > 0$ in an interval $[x_1, x_1 + d]$ for some $d > 0$. If u_g denotes the solution of (BVP) such that $f(x) = h(t) = 0$ and (8.1) holds then there exist a $\Delta t > 0$ and a constant $K \geq 0$ such that

$$|u_g(x,t) - W^{\#N}_{\gamma, x_1}(x,t)| \leq K \sqrt{t}^{\gamma+N+\alpha} \tag{8.3}$$

uniformly in $\bar{D}$. Similarly if (A) holds with $N + \alpha > 0$ in $[x_2 - d, x_2]$ for some $d > 0$ and if u_h denotes the solution of (BVP) such that $f(x) = 0$, $g(t) = 0$ and (8.2) holds, then

$$|u_h(x,t) - W^{N}_{\gamma, x_2}(x,t)| \leq K \sqrt{t}^{\gamma+N+\alpha} \tag{8.4}$$

uniformly in $\bar{D}$.

PROOF. See Theorem (3.3.1) of Reference 1.

The intuitive reasoning behind this result is, first, that by their very construction the boundary expansions will accurately approximate u_g and u_h near the respective boundaries $x = x_1$ and $x = x_2$, and secondly, that these expansions, and these derivatives, decay exponentially away from the first boundary and thereby effectively satisfy the homogeneous equation (2.1) and vanish at the opposite boundary. The theorem is established by incorporating these ideas into a formal application of the maximum principle for parabolic differential equations.

Theorem (8.1) can clearly be generalized to include functions $g(t)$ and $h(t)$ which are finite linear combinations of functions $h_{\gamma/2}(t)$. Furthermore, if $g(t)$ (or $h(t)$) possesses a jump discontinuity in the form

$$g(t) = h_{\gamma/2}(t-t_o) = \begin{cases} 0 & 0 \le t \le t_o \\ \sqrt{t-t_o}^{\,\gamma}/(\gamma/2)! & t > t_o \end{cases}$$

where $0 \le t \le \Delta t$ then u_g can be approximated by the function

$$u_g \approx \begin{cases} 0 & 0 \le t \le t_o \\ W^{\#N}_{\gamma,x}(x,t-t_o) & t_o \le t \le \Delta t \end{cases}$$

Thus, Theorem (8.1) can be directly extended to include any piecewise continuous boundary values of the form

$$\left.\begin{array}{c} g(t) \\ \text{or} \\ h(t) \end{array}\right\} = \sum_{k=1}^{K} \alpha_k \, h_{\gamma_k/2}(t-t_k) + e(t) \qquad (8.5)$$

where $0 \le \gamma \le N$ and $e(t)$ is an error term such that

$$|\,e(t)\,| \le K \sqrt{t}^{\,N+\alpha}$$

for some constant $K > 0$. In this more general case however the term on the right hand sides of (8.3) and (8.4) should be replaced by

$$K \sqrt{t}^{\,N+\alpha}$$

since the lowest possible value of γ, $\gamma = 0$, must be used. We shall not formulate an additional theorem to express these results since they are straightforward consequences of Theorem (8.1). This concludes our treatment of problems $(BVP)_g$ and $(BVP)_h$ and we turn now to problem $(BVP)_f$.

The regular and special expansions which we applied to the Cauchy problem in Section VII are still meaningful in the context of problem (BVP) , provided $f(x)$ and the coefficients a, b and c are sufficiently differentiable. In fact, for sufficiently small times, these expansions will accurately approximate the solution of (BVP) except in regions close to the boundaries $x = x_1, x_2$. Theorems (7.2) and (7.3) of the last section remain valid with only minor modifications and may be stated as follows:

[redacted] Let $I \subset (x_1, x_2)$ be an open interval, let (A) hold in I with $N + \alpha > 1$ and let $f \in C^{N+\alpha}(I)$. If u_f denotes the solution of problem (BVP) when $g(t) = h(t) = 0$ and if

$$U^{\bar{N}}(x,t) = \sum_{k=0}^{\bar{N}} L^{(k)} f(x)\, t^k/k! \qquad (8.6)$$

where $\bar{N} = [N/2]$ then there exists a $\Delta t > 0$ and for each compact subset $S \subset I$ there exists a constant K such that

$$|u_f(x,t) - U^{\bar{N}}(x,t)| \leq K \sqrt{t}^{\,N+\alpha} \tag{8.7}$$

uniformly in $S \times [o,\Delta t]$.

PROOF. See Theorem (3.2.4) of Reference 1.

THEOREM Let $I \subset (x_1, x_2)$ be an open interval, let (A) hold in I with $N + \alpha > 0$ and let $x_0 \in I$. Let u_f denote the solution of (BVP) such that $g(t) = h(t) = 0$ and suppose that

$$f(x) = h_\gamma(x - x_o)$$

for some $\gamma \geq 0$ and $x_1 \leq x \leq x_2$. If $\gamma + N + \alpha > 1$ then there exist a constant $K > 0$ and a $\Delta t > 0$ such that

$$|u_f(x,t) - V^N_{\gamma,x_o}(x,t)| \leq K \sqrt{t}^{\,\gamma+N+\alpha} \tag{8.8}$$

uniformly in $[x_1, x_0] \times [o,\Delta t]$. Similarly if

$$f(x) = h^*_\gamma(x-x_o)$$

then

$$|u_f(x,t) - V^{\#N}_{\gamma,x_o}(x,t)| \leq K \sqrt{t}^{\,\gamma+n+\alpha} \tag{8.9}$$

uniformly in $[x_0, x_2] \times [o,\Delta t]$. Here the functions V^N_{γ,x_o} and $V^{\#N}_{\gamma,x_o}$ are defined by (6.1) and (6.5)

PROOF. See Theorem (3.2.5) and (3.3.3) of Reference 1.

By combining Theorems (8.2) and (8.3) we can obtain a result, for problem $(BVP)_f$, almost identical to Theorem (7.5) in which initial values of the form

$$f(x) = \bar{f}(x) + \sum_{k=1}^{K} \sum_{j=0}^{N} \alpha_{kj} h_j(x-x_k)$$

are treated, where $\bar{f} \in C^{N+\alpha}(I)$ with $I \subset (x_1, x_2)$. The only modification would be that the restrictions on N, α and γ would now be N, α, $\gamma \geq 0$ with $N + \alpha > 1$. We omit a formal statement of this Theorem.

Our analysis is still not complete however since we have little information about the behavior of U near the boundaries $x = x_1, x_2$, even when the open interval in Theorem (8.2) and (8.3) can be taken as $I = (x_1, x_2)$. The dependence of the constant K on the compact subset $S \subset I$ is a critical weakness since it means that no single constant K exists such that inequalities (8.7) and (8.8) hold uniformly in all of D.

Let us consider, therefore, how to improve our approximations in the vicinity of $x = x_2$.

Since the closest discontinuity in the initial values $f(x)$ (if any exists) occurs at some finite distance away from $x = x_2$ then we may suppose that $f(x)$ is smooth, say of class $C^{N+\alpha}$ in some neighborhood of $x = x_2$. The regular expansion (8.6) is thus well defined for x in this region but cannot approximate u_f accurately since along $x = x_2$ it assumes the (in general) non-vanishing values

$$U^{\bar{N}}(x_2,t) = \sum_{k=0}^{\bar{N}} L^{(k)} \; f(x_2) \; t^k/k!$$

In effect $U^{\bar{N}}$ is approximating the solution of the wrong problem, namely one for which

$$h(t) = \sum_{k=o}^{\bar{N}} L^{(k)} f(x_2) \quad t^k/k! \tag{8.10}$$

along $x = x_2$ instead of $h(t) = 0$. To counteract this it would seem natural to adjust $U^{\bar{N}}$ by subtracting away the solution of problem $(BVP)_h$ for which $h(t)$ is given by (8.10). But in this form $h(t)$ is simply a finite power series in t so the methods already developed for problem $(BVP)_h$ apply and lead us to define the following <u>N-th order right hand boundary correction for problem $(BVP)_f$</u>

$$w^N(x,t) \equiv \sum_{k=0}^{\bar{N}} L^{(k)} f(x_2) \; W^{N-2k}_{2k,x_2} \quad (x,t) \tag{8.11}$$

Similarly, we define the <u>N-th order left hand boundary correction for $(BVP)_f$</u> by

$$w^{\#N}(x,t) \equiv \sum_{k=0}^{\bar{N}} L^{(k)} f(x_1) \; W^{\#N-2k}_{2k,x_1} \quad (x,t) \tag{8.11\#}$$

We can now state the following theorem:

[redacted] Let $N, \alpha \geq 0$ with $N + \alpha > 1$, let (A) hold in (x_1, x_2) and let a, b, c be of class $C^{N+\alpha}$ in $[x_1, x_1 + d)$ and $(x_2 - d, x_2]$ for some $d > 0$. If $f \in C^{N+\alpha}([x_1, x_2])$ and if u_f denotes the solution of (BVP) when $g(t) = h(t) = 0$ then there exist a $\Delta t > 0$ and a constant $K \geq 0$ such that

$$|(u_f - U^{\bar{N}} + w^N + w^{\#N}) \; (x,t)| \leq K \sqrt{t}^{N+\alpha} \tag{8.12}$$

uniformly in $\overline{D} = [x_1, x_2] \times [0, \Delta t]$.

PROOF: See Theorem (3.3.6) of Reference 1.

This essentially completes our discussion. Uniformly valid approximations for u_f when

$$f(x) = h_\gamma (x - x_o)$$

or

$$f(x) = h^*_\gamma (x - x_o)$$

in the cases where $\gamma = n$, a non-negative integer, can be obtained by repeating the device of Section VII involving identity (7.5) and applying Theorems (8.3) and (8.4).

Thus, to summarize this section, we can construct approximate solutions to problem (BVP) which, for sufficiently small $\Delta t > 0$, are accurate to order $\Delta t^{N+\alpha}$ uniformly in D whenever g(t) and h(t) have the general form (8.5) and

$$f(x) = \overline{f}(x) + \sum_{k=1}^{K} \sum_{j=0}^{N} \alpha_{kj} h_j (x - x_k)$$

where $\overline{f} \in C^{N+\alpha}([x_1, x_2])$. Each of the terms in these forms can be treated separately because of the linearity of problem (BVP) and the general result obtained by superposition.

APPENDIX - THE COEFFICIENTS $\alpha_{\gamma,n,k}$ AND $\beta_{\gamma,n,k}$

The coefficients $\alpha_{\gamma,n,k}$ and $\beta_{\gamma,n,k}$ appearing in (5.7) and (5.15) respectively can be obtained from formulae (5.8) and (5.16) respectively. Since these become somewhat laborious to compute explicitly as n increases we include here the results for cases n = 0,1,2. For larger n it would seem advisable to utilize a symbol manipulation program to double check the results.

In the following formulae let

$$\bar{a}_i = a_i/a_o \qquad i = 0,1,2$$

$$\bar{b}_i = b_i/a_o \qquad i = 0,1$$

$$\bar{c}_i = c_i/a_o \qquad i = 0$$

n = 0

$$\alpha_{\gamma,0,0} = 1$$

$$\beta_{\gamma,0,0} = 1$$

n = 1

$$\alpha_{\gamma,1,0} = \frac{1}{4}(\gamma^2-1)\bar{a}_1 + \frac{1}{2}(\gamma+1)\bar{b}_o$$

$$\alpha_{\gamma,1,1} = \frac{1}{4}\bar{a}_1 - \frac{1}{2}\bar{b}_o$$

$$\alpha_{\gamma,1,2} = -\frac{1}{2}\bar{a}_1$$

$$\beta_{\gamma,1,0} = 0$$

$$\beta_{\gamma,1,1} = \frac{1}{4}\bar{a}_1 - \frac{1}{2}\bar{b}_o$$

$$\beta_{\gamma,1,2} = -\frac{1}{2}\bar{a}_1$$

n = 2

$$\alpha_{\gamma,2,0} = \frac{1}{24}\gamma(\gamma+2)(2\gamma-1)\bar{a}_2 + \frac{1}{4}\gamma(\gamma+2)\bar{b}_1 + \frac{1}{2}\gamma(\gamma+2)\bar{c}_o$$
$$+ \frac{1}{32}\gamma^2(\gamma^2-4)\bar{a}_1^2 + \frac{1}{8}\gamma(\gamma-1)(\gamma+2)\bar{a}_1\bar{b}_o + \frac{1}{8}\gamma(\gamma+2)\bar{b}_o^2$$

$$\alpha_{\gamma,2,1} = -\frac{1}{8}\bar{a}_2 + \frac{1}{4}\bar{b}_1 - \frac{1}{2}c_o + \frac{1}{32}(2\gamma^2+1)\bar{a}_1^2 - \frac{1}{8}\gamma(\gamma-1)\bar{a}_1\bar{b}_o - \frac{1}{8}(2\gamma+1)\bar{b}_o^2$$

$$\alpha_{\gamma,2,2} = \frac{1}{4}\bar{a}_2 - \frac{1}{2}\bar{b}_1 - \frac{1}{16}(2\gamma^2+1)\bar{a}_1^{\,2} - \frac{1}{4}\gamma\bar{a}_1\bar{b}_o + \frac{1}{4}\bar{b}_o^2$$

$$\alpha_{\gamma,2,3} = -\frac{1}{2}\bar{a}_2 + \frac{3}{4}\bar{a}_1\bar{b}_o + \frac{3}{8}\bar{a}_1^{\,2}$$

$$\alpha_{\gamma,2,4} = \frac{3}{4}\bar{a}_1^{\,2}$$

$$\beta_{\gamma,2,0} = 0$$

$$\beta_{\gamma,2,1} = -\frac{1}{8}\bar{a}_2 + \frac{1}{4}\bar{b}_1 - \frac{1}{2}\bar{c}_o + \frac{3}{32}\bar{a}_1^2 - \frac{1}{4}\bar{a}_1\bar{b}_o + \frac{1}{8}\bar{b}_o^{\,2}$$

$$\beta_{\gamma,2,2} = \frac{1}{2}\bar{a}_2 - \frac{1}{2}\bar{b}_1 - \frac{3}{16}\bar{a}_1^2 + \frac{1}{4}\bar{a}_1\bar{b}_o + \frac{1}{4}\bar{b}_o^{\,2}$$

$$\beta_{\gamma,2,3} \equiv -\frac{1}{2}\bar{a}_2 + \frac{3}{8}\bar{a}_1^{\,2} + \frac{3}{4}\bar{a}_1\bar{b}_o$$

$$\beta_{\gamma,2,4} = \frac{3}{4}\bar{a}_1^{\,2}$$

For completeness we also list the intermediate coefficients $\bar{\alpha}_{\gamma,n,k}$ and $\bar{\beta}_{\gamma,n,k}$, $n = 1,2$, appearing in equations (5.9) and (5.17) respectively:

$n = 1$

$$\bar{\alpha}_{\gamma,1,0} = b_o$$

$$\bar{\alpha}_{\gamma,1,1} = a_1$$

$$\bar{\beta}_{\gamma,1,0} = b_o$$

$$\bar{\beta}_{\gamma,1,1} = a_1$$

$n = 2$

$$\bar{a}_{\gamma,2,0} = a_o[\frac{1}{4}\gamma^2\bar{a}_1\bar{b}_o + \frac{1}{2}\gamma\bar{b}_o^2 + \bar{c}_o]$$

$$\bar{a}_{\gamma,2,1} = a_o[\frac{1}{4}(\gamma^2-1)\bar{a}_1^{\,2} - \frac{1}{2}\bar{b}_o^2 + \frac{1}{4}(2\gamma-3)\bar{a}_1\bar{b}_o+\bar{b}_1]$$

$$\bar{\alpha}_{\gamma,2,2} = a_o[-\frac{3}{2}\bar{a}_1^{\,2} - \frac{3}{2}\bar{a}_1\bar{b}_o + \bar{a}_2]$$

$$\bar{\alpha}_{\gamma,2,3} = a_o[-\frac{3}{2}\bar{a}_1^{\,2}]$$

$$\beta_{\gamma,2,0} = a_o [\bar{c}_o + \frac{1}{4} \bar{a}_1 \bar{b}_o - \frac{1}{2} \bar{b}_o^{\,2}]$$

$$\beta_{\gamma,2,1} = a_o [\bar{b}_1 - \frac{5}{4} \bar{a}_1 \bar{b}_o - \frac{1}{2} \bar{b}_o^{\,2}]$$

$$\beta_{\gamma,2,2} = a_o [\bar{a}_2 - \frac{3}{2} \bar{a}_1^{\,2} - \frac{3}{2} \bar{a}_1 \bar{b}_o]$$

$$\beta_{\gamma,2,3} = a_o [- \frac{3}{2} \bar{a}_1^{\,2}]$$

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